

ID: \_\_\_\_\_ Name: \_\_\_\_\_

滿分 20 分, 每題 4 分

(1)

$$f(t) = \begin{cases} 1, & 0 \leq t < 4 \\ 0, & 4 \leq t < 5 \\ 1, & t \geq 5 \end{cases}$$

(2)

$$L^{-1}\left\{\frac{(1 + e^{-2s})^2}{s + 2}\right\}$$

(3)

$$F(s) = \frac{e^{-2s}}{(s - 1)^3} \quad \text{Find } f(t)$$

(4) find  $b_n$

$$f(x) = x + \pi, \quad -\pi < x < \pi \quad \text{and} \quad f(x + 2\pi) = f(x)$$

(5) find the Fourier series of the function  $f$  on the given interval

$$f(x) = \begin{cases} 2+x, & -2 < x < 0 \\ 2, & 0 \leq x < 2 \end{cases}$$

## 公式表:

- Laplace Transform

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$	
1	1	$1/s$	7	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	First Shifting Theorem, $s$ -Shifting
2	$t$	$1/s^2$	8	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$
3	$t^2$	$2!/s^3$	9	$\cosh at$	$\frac{s}{s^2 - a^2}$	$e^{at}f(t) = \mathcal{L}^{-1}\{F(s - a)\}$
4	$t^n$ ( $n = 0, 1, \dots$ )	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2 - a^2}$	Laplace Transform of Derivatives
5	$t^a$ ( $a$ positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\frac{s - a}{(s - a)^2 + \omega^2}$	$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$
6	$e^{at}$	$\frac{1}{s - a}$	12	$e^{at} \sin \omega t$	$\frac{\omega}{(s - a)^2 + \omega^2}$	$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$
						$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$
						Laplace Transform of Integral 積分轉換
						$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} F(s), \quad \text{thus} \quad \int_0^t f(\tau) d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s} F(s)\right\}$

### Second Shifting Theorem; Time Shifting

$$\tilde{f}(t) = f(t - a)u(t - a) = \begin{cases} 0 & \text{if } t < a \\ f(t - a) & \text{if } t > a \end{cases}$$

$$\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as}F(s)$$

$$f(t - a)u(t - a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}$$

### Examples for $t$ -shifting by 1

$$1 \mapsto \frac{1}{s} \quad u(t-1) \mapsto e^{-s} \frac{1}{s}$$

$$t \mapsto \frac{1}{s^2} \quad (t-1)u(t-1) \mapsto e^{-s} \frac{1}{s^2}$$

$$t^2 \mapsto \frac{2}{s^3} \quad (t-1)^2 u(t-1) \mapsto e^{-s} \frac{2}{s^3}$$

$$\sin t \mapsto \frac{1}{s^2 + 1} \quad \sin(t-1)u(t-1) \mapsto e^{-s} \frac{1}{s^2 + 1}$$

### 傅立葉級數 (週期 $2L$ )

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

(a)  $a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$

(b)  $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \quad n = 1, 2, \dots$

(c)  $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \quad n = 1, 2, \dots$

### Fourier Cosine Series (even function, 偶函数) (週期 $2L$ )

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \quad (f \text{ even})$$

$$a_0 = \frac{1}{L} \int_0^L f(x) dx, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 1, 2, \dots$$

### Fourier Sine Series (odd function, 奇函数) (週期 $2L$ )

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad (f \text{ odd})$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$\sin \pi = 0, \quad \sin 2\pi = 0, \quad \sin 3\pi = 0, \dots$$

$$\cos \pi = -1, \quad \cos 2\pi = 1, \quad \cos 3\pi = -1, \dots$$